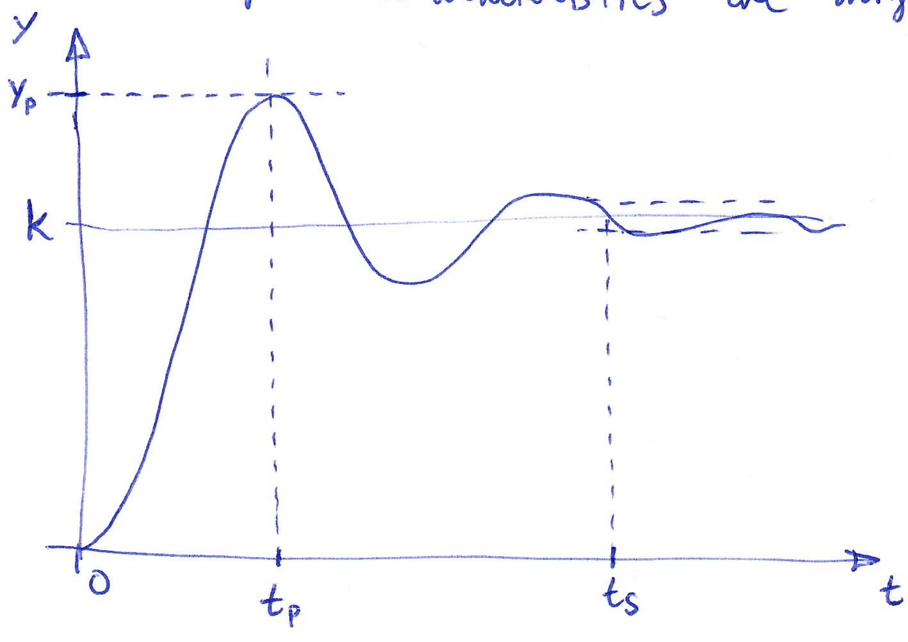
ME 4555 - Lecture 20 - more second order systems

Recall the step response for an underdamped $(0 < \zeta < 1)$ system. The transfer function is: $G(s) = \dfrac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

and the step response is: $y(t) = K\left[1 - \dfrac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)\right]$

where: $\begin{cases} K = \text{system gain}, \\ \omega_n = \text{natural frequency}, \\ \zeta = \text{damping ratio}, \end{cases}$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)
$\phi = \arccos(\zeta)$ (pole angle).

Some useful characteristics we might want to control:



t_p: "peak time". Time to reach max height.

PO: "percent overshoot"

$$PO = \frac{y_p - k}{k} \times 100\%$$

t_s: "settling time": Time until we stay within some bound of the final value (usually 2% is used)

Textbooks also often define "rise time" (also in Matlab); it is similar to peak time. Matlab defines it as the time to go from 0.1K to 0.9K. We will mainly discuss t_p, PO, and t_s.

Peak time (t_p).

Peak time occurs when $y'(t) = 0$. (max occurs at zero derivative).
we can calculate this directly:

$$y(t) = K\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)\right]$$

$$\Rightarrow y'(t) = \frac{-K}{\sqrt{1-\zeta^2}}\left(-\zeta\omega_n e^{-\zeta\omega_n t} \sin(\omega_d t+\phi) + \omega_d e^{-\zeta\omega_n t} \cos(\omega_d t+\phi)\right)$$

setting equal to zero:

$$-\zeta\omega_n \sin(\omega_d t + \phi) + \omega_d \cos(\omega_d t + \phi) = 0$$

use fact that $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\zeta = \cos\phi$, $\sqrt{1-\zeta^2} = \sin\phi$.

$$\Rightarrow -\cos\phi \sin(\omega_d t + \phi) + \sin\phi \cos(\omega_d t + \phi) = 0$$

$$\Rightarrow \sin(\omega_d t + \phi - \phi) = 0$$

$$\Rightarrow \sin(\omega_d t) = 0$$

So $y'(t) = 0$ occurs at $t = \left\{0, \frac{\pi}{\omega_d}, \frac{2\pi}{\omega_d}, \dots\right\}$.

first peak!

Therefore $\boxed{t_p = \frac{\pi}{\omega_d}}$

If we want a faster response (peak occurs earlier), we should make t_p small, so ω_d should be large.

Percent overshoot (PO)

Percent overshoot is how much higher the first peak is compared to the final value. The formula is:

$$PO = 100 \cdot \frac{y(t_p) - K}{K} \qquad \left(\text{since peak occurs at } t_p\right)$$

Substituting in our formula for PO using y and t_p,

$$PO = 100 \cdot \frac{-1}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t_p} \sin\left(\omega_d t_p + \phi\right)$$

$$= \frac{100}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} \sin(\phi)$$

use fact that:

$\cos\phi = \zeta$

$\sin\phi = \sqrt{1-\zeta^2}$

$\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$

$$\boxed{PO = 100\, e^{-\pi/\tan\phi}}$$

Percent overshoot is determined by the angle ϕ.

Small $\phi \Leftrightarrow$ small $\tan\phi \Leftrightarrow$ large $\frac{\pi}{\tan\phi} \Leftrightarrow$ small PO.

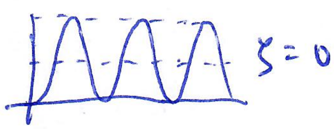
So if we want a <u>smaller</u> PO, we should make ϕ smaller.

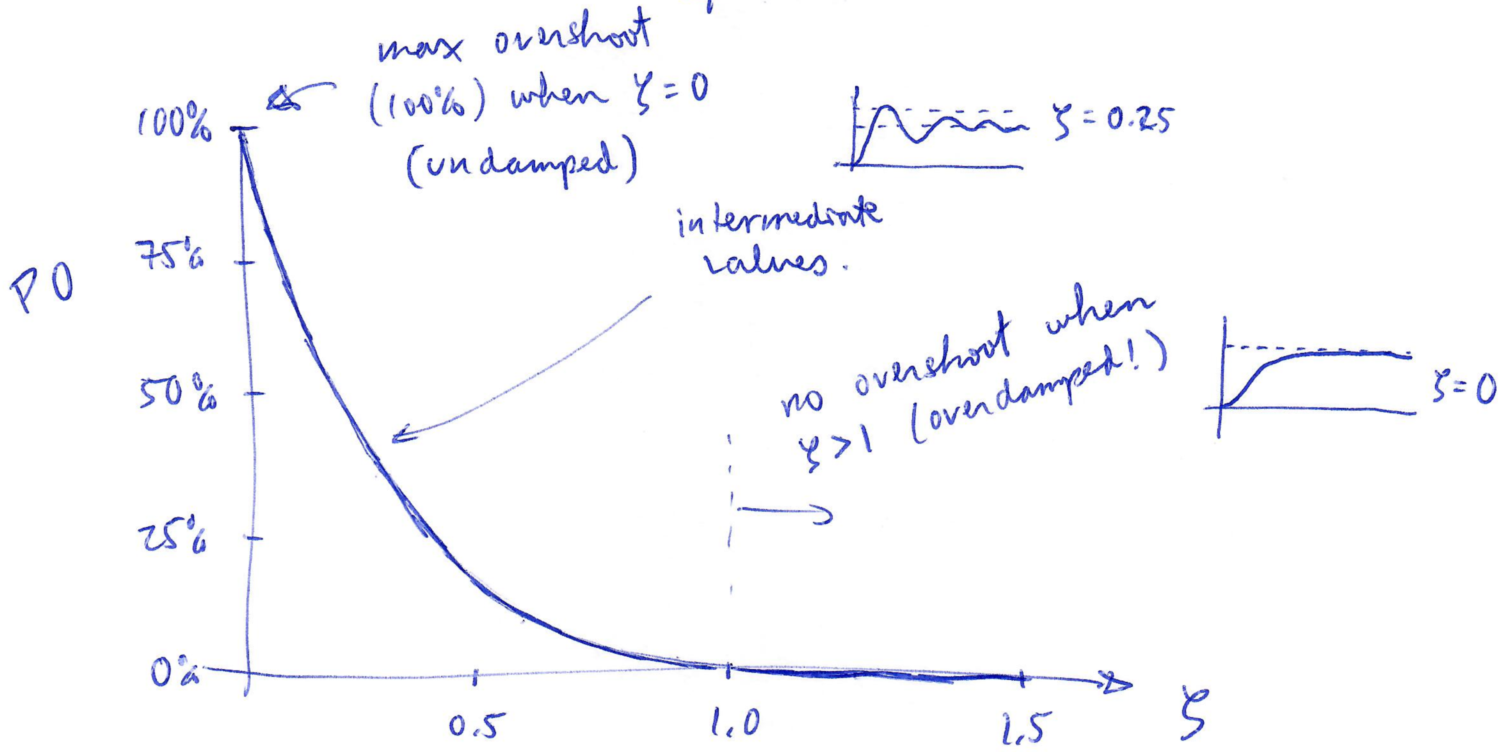
Note: Percentage overshoot can also be calculated directly from ζ:

$$PO = 100\, e^{-\pi \zeta / \sqrt{1-\zeta^2}}$$

This formula is tedious to write out / work with, so one can use a handy reference such as a ready-made plot to look up values.

Looks like this:





Sometimes percent overshoot (PO) is written as "maximum overshoot" (M_p). These mean exactly the same thing.

Settling time (t_s)

Settling time tells us when we are close to our final value and will remain close. This has to do with the exponential envelope $e^{-\zeta\omega_n t}$ found in $y(t)$. For this exponential, $\tau = \frac{1}{\zeta\omega_n}$ is the time constant. So if we want to be within 2% of the final value, a good approximation is 4τ.

$$\boxed{t_s \approx \frac{4}{\zeta\omega_n}}$$

For a faster settling time (smaller t_s), we should make $\zeta\omega_n$ as large as possible.

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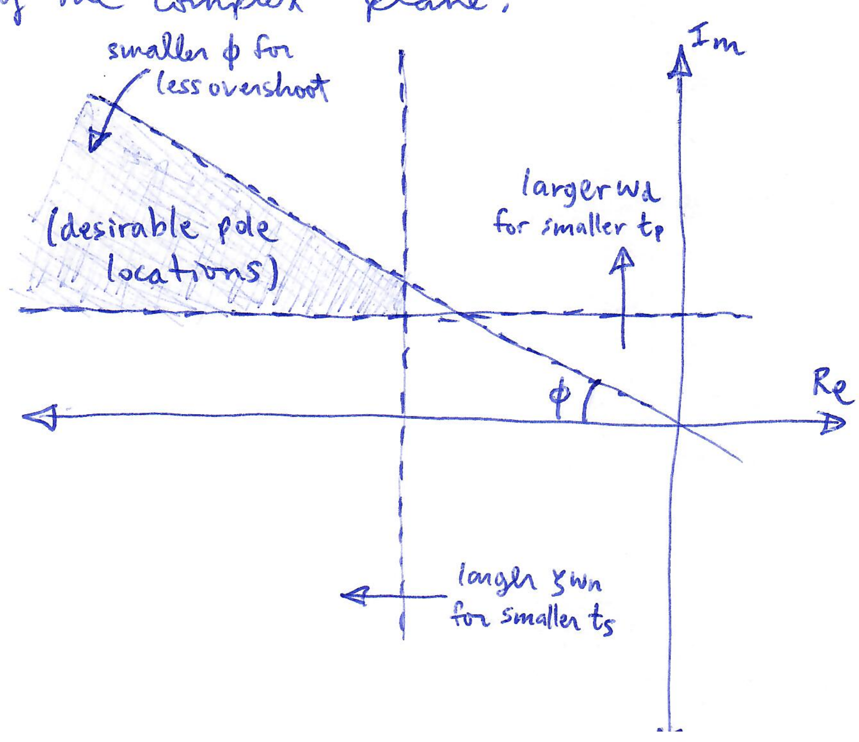
t_p, PO, t_s have a geometric interpretation:

Our poles are at $-\zeta\omega_n \pm \omega_d i$.

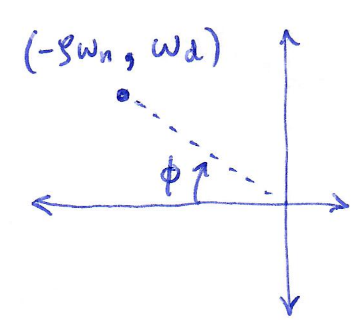
Each characteristic corresponds to a region of the complex plane:



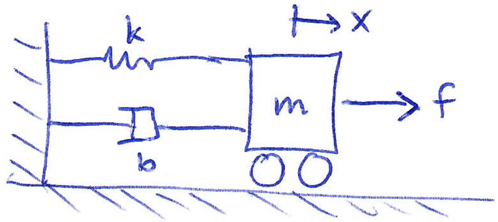
t_s depends on $\zeta\omega_n$ (real part of pole)

t_p depends on ω_d (imaginary part of pole)

PO depends on ϕ (pole angle)



Ex : mass-spring-damper system



We have: $m = 1\,kg$, $b = 1\,N{\cdot}s/m$, $k = 10\,N/m$.

* find ω_n, ξ, K, ω_d, ϕ.
* find t_p, t_s, PO
* sketch step response

Transfer function: $\dfrac{X}{F} = \dfrac{1}{ms^2 + bs + k} = \dfrac{1}{s^2 + s + 10} = \dfrac{1}{10}\left(\dfrac{10}{s^2 + 2\cdot\frac{1}{2\sqrt{10}}\cdot\sqrt{10}s + 10}\right)$

therefore: $\omega_n = \sqrt{10} \approx 3.16$ rad/sec ≈ 0.5 Hz (undamped frequency)

$\xi = \dfrac{1}{2\sqrt{10}} \approx 0.158$ (underdamped)

$K = \dfrac{1}{10} = 0.1$ (final value)

$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.12$ rad/sec ≈ 0.496 Hz (damped frequency).

$\phi = \arccos(\xi) \approx 1.412$ rad $= 80.9°$ (pole angle).

Step response is: $\boxed{x(t) = 0.1\left(1 - 1.01\,e^{-0.5t}\sin(3.12t + 1.412)\right)}$

$\boxed{t_p = \dfrac{\pi}{\omega_d} \approx 1.01 \text{ sec}}$. $\boxed{PO = 100\,e^{-\pi/\tan\phi} \approx 60.5\%}$ $\boxed{t_s = \dfrac{4}{\xi\omega_n} \approx 8.0 \text{ sec.}}$

Sketch: